

SYMMETRIC POWERS OF SEVERI-BRAUER VARIETIES

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ABSTRACT. We classify products of symmetric powers of a Severi-Brauer variety, up-to stable birational equivalence. The description also includes Grassmannians and moduli spaces of genus 0 stable maps.

There are several ways to associate other varieties to a Severi-Brauer variety P defined over a field k . These include

- the Grassmannians $\text{Grass}(\mathbb{P}^{m-1}, P)$,
- the symmetric powers $\text{Sym}^m(P)$ and
- the moduli spaces $\bar{M}_0(P, d)$ of genus 0 stable maps of degree d to P .

While all these varieties are geometrically rational, they are usually not rational over the ground field and it is an interesting problem to understand their birational properties over k . The results of this note are partly weaker—since we describe only the stable birational equivalence classes—but partly stronger—since we also describe the products of these varieties.

Thus let $\text{MSym}(P)$ denote the multiplicative monoid generated by stable birational equivalence classes of Grassmannians of P , symmetric powers of P and the moduli spaces $\bar{M}_0(P, d)$. We show that $\text{MSym}(P)$ is finite, identify its elements and also the multiplication rules.

Birational equivalence of two varieties is denoted by $X \stackrel{\text{bir}}{\sim} Y$ and stable birational equivalence by $X \stackrel{\text{stab}}{\sim} Y$. See Paragraph 2 for the definition and basic properties of Severi-Brauer varieties.

Theorem 1. *Let P be a Severi-Brauer variety of index $i = i(P)$. Then*

- (1) $\text{MSym}(P) = \{\text{Grass}(\mathbb{P}^{d-1}, P) : d \mid i(P)\}$ and products are given by
- (2) $\text{Grass}(\mathbb{P}^{d-1}, P) \times \text{Grass}(\mathbb{P}^{e-1}, P) \stackrel{\text{stab}}{\sim} \text{Grass}(\mathbb{P}^{(d,e)-1}, P)$, where (d, e) denotes the greatest common divisor. The identity is $\text{Grass}(\mathbb{P}^{i-1}, P) \stackrel{\text{stab}}{\sim} \mathbb{P}^0$.

The class of an arbitrary Grassmannian is given by the rule

- (3) $\text{Grass}(\mathbb{P}^{d-1}, P) \stackrel{\text{stab}}{\sim} \text{Grass}(\mathbb{P}^{(d,i)-1}, P)$.

The class of an arbitrary symmetric power is given by the rules

- (4) $\text{Sym}^d(P) \stackrel{\text{stab}}{\sim} \text{Sym}^{(d,i)}(P)$ for every $d \geq 0$,
- (5) $\text{Sym}^d(P) \stackrel{\text{bir}}{\sim} \text{Grass}(\mathbb{P}^{d-1}, P) \times \mathbb{P}^{d(d-1)}$ for $d \leq n+1$ and
- (6) $\text{Sym}^d(P) \times \text{Sym}^e(P) \stackrel{\text{stab}}{\sim} \text{Sym}^{(d,e)}(P)$.

The class of $\bar{M}_0(P, d)$ is determined by the parity of d :

- (7) $\bar{M}_0(P, 2e) \stackrel{\text{stab}}{\sim} P$ save when $\dim P = e = 1$ and
- (8) $\bar{M}_0(P, 2e+1) \stackrel{\text{stab}}{\sim} \text{Grass}(\mathbb{P}^1, P)$. Note that $\text{Grass}(\mathbb{P}^1, P)$ is rational iff $i(P) \in \{1, 2\}$ and stably birational to P iff $i(P)$ is odd.

The most natural description seems to be in terms of symmetric powers, so we start with them. The relationship with Grassmannians is easy to establish. The

moduli spaces $\bar{M}_0(P, d)$ end up birationally the simplest but understanding them is more subtle.

The case $\dim P = e = 1$ is exceptional in (1.7). $\bar{M}_0(P, 2)$ aims to classify double covers of P ramified at 2 points. The coarse moduli space is $\text{Sym}^2(P) \cong \mathbb{P}^2$. However, if $P \not\cong \mathbb{P}^1$ then there are no such double covers defined over k . The problem is that every double cover has an order 2 automorphism. In all other cases, a dense open subset of $\bar{M}_0(P, d)$ parametrizes maps without automorphisms, even embeddings if $\dim P \geq 3$.

2 (Severi-Brauer varieties I). Let k be a field with separable closure k^s . A k -scheme P is called a *Severi-Brauer variety* if $P_{k^s} := P \times_{\text{Spec } k} \text{Spec } k^s \cong \mathbb{P}^n$ for some n . We say that P is trivial if $P \cong \mathbb{P}^n$. The *index* of P is the gcd of all 0-cycles on P ; it is denoted by $i(P)$. A subscheme $L \subset P$ is called *twisted linear* if L_{k^s} is a linear subspace of $P_{k^s} \cong \mathbb{P}^n$. Thus L is also a Severi-Brauer variety. For a reduced subscheme $Y \subset X$ let $\langle Y \rangle$ denote the minimal twisted linear subvariety containing Y . Thus $\langle Y \rangle_{k^s}$ is the linear span of Y_{k^s} .

The following basic results go back to Severi and Châtelet, see [GS06, Chap.5] for a modern treatment and references.

- (1) P is trivial iff $P(k) \neq \emptyset$.
- (2) $i(P)$ divides $\dim P + 1$,
- (3) P has a 0-cycle Z of degree $i(P)$ and $\dim \langle Z \rangle = i(P) - 1$.
- (4) The minimal twisted linear subvarieties have dimension $i(P) - 1$ and they are isomorphic to each other; call this isomorphism class P^{\min} .
- (5) Given P^{\min} and $r \geq 1$ there is a unique (up-to isomorphism) Severi-Brauer variety P_r of dimension $r(\dim P^{\min} + 1) - 1$ such that $(P_r)^{\min} \cong P^{\min}$.
- (6) $P_r \stackrel{\text{bir}}{\sim} P^{\min} \times \mathbb{P}^m$ for $m = (r - 1)(\dim P^{\min} + 1)$.
- (7) Two Severi-Brauer varieties P_1, P_2 are *Brauer-equivalent*, denoted by $P_1 \sim P_2$, iff $P_1^{\min} \cong P_2^{\min}$. This holds iff the smaller dimensional one is isomorphic to a twisted linear subvariety of the other.

1. SYMMETRIC POWERS

A key step in understanding symmetric powers is the following.

Theorem 3. [KS04] *Let P be a Severi-Brauer variety of dimension n . Then $\text{Sym}^{n+1}(P)$ is rational.*

The following is a short geometric proof. The Euler number of \mathbb{P}^n is $n + 1$, thus a general section of the tangent bundle $T_{\mathbb{P}^n}$ vanishes at $n + 1$ points. For any Severi-Brauer variety this gives a dominant map $\pi : H^0(P, T_P) \dashrightarrow \text{Sym}^{n+1}(P)$.

Let $Z \subset P$ be a reduced 0-cycle of degree $n + 1$. Then $\pi^{-1}(Z)$ is the linear space $H^0(P, T_P(-Z)) \subset H^0(P, T_P)$ of dimension $n + 1$. Let $V \subset H^0(P, T_P)$ be a general affine-linear subspace of codimension $n + 1$. Then $\pi|_V : V \dashrightarrow \text{Sym}^{n+1}(P)$ is birational. \square

Corollary 4. *Let P be a Severi-Brauer variety of index $i(P)$. Then $\text{Sym}^d(P)$ is stably rational iff $i(P) \mid d$.*

Proof. If d is not divisible by $i(P)$ then $\text{Sym}^d(P)(k) = \emptyset$, hence $\text{Sym}^d(P)$ is not stably rational.

To see the converse, assume that $i(P) \mid d$. By (2.5–6) P is stably birational to a Severi-Brauer variety P' of dimension $d - 1$. Furthermore, $\text{Sym}^d(P')$ is rational by Theorem 3 and it is stably birational to $\text{Sym}^d(P)$ by Corollary 8. \square

5 (Proof of Theorem 1.1–6). The easiest is (1.5). Given d points in general position, they span a linear subspace of dimension $d - 1$. This gives a natural map $\pi : \text{Sym}^d(P) \dashrightarrow \text{Grass}(\mathbb{P}^{d-1}, P)$. Let K be the function field of $\text{Grass}(\mathbb{P}^{d-1}, P)$ and $L_K \subset P_K$ the linear subspace corresponding to the generic point. Thus L_K is a Severi-Brauer subvariety of dimension $d - 1$. The generic fiber of π is $\text{Sym}^d(L_K)$ which is rational by Theorem 3. Thus $\text{Sym}^d(P) \stackrel{\text{bir}}{\sim} \text{Grass}(\mathbb{P}^{d-1}, P) \times \mathbb{P}^{d(d-1)}$.

Next we show (1.6) using the stable birational equivalences

$$\begin{aligned} \text{Sym}^d(P) \times \text{Sym}^e(P) \times \mathbb{P}^{n(d,e)} &\stackrel{\text{stab}}{\sim} \text{Sym}^d(P) \times \text{Sym}^e(P) \times \text{Sym}^{(d,e)}(P) \\ &\stackrel{\text{stab}}{\sim} \mathbb{P}^{nd} \times \mathbb{P}^{ne} \times \text{Sym}^{(d,e)}(P). \end{aligned}$$

First let K be the function field of $\text{Sym}^d(P) \times \text{Sym}^e(P)$. Then P_K has 0-cycles of degrees d and e , thus is also has a 0-cycle of degree (d, e) . Thus $\text{Sym}^{(d,e)}(P_K)$ is stably rational by Corollary 4, proving the first part.

Similarly, let L be the function field of $\text{Sym}^{(d,e)}(P)$. Then $\text{Sym}^d(P_L)$ and $\text{Sym}^e(P_L)$ are stably rational by Lemma 4, proving the second part.

Using this for $e = i(P)$ gives that

$$\text{Sym}^d(P) \times \text{Sym}^i(P) \stackrel{\text{stab}}{\sim} \text{Sym}^{(d,i)}(P).$$

Since $\text{Sym}^i(P)$ is stably rational by Corollary 4, we (1.4). Together with (1.5) this implies (1.2).

We have proved that every class in $\text{MSym}(P)$ is stably birational to a symmetric power $\text{Sym}^d(P)$ for some $d \mid i(P)$. Next we show that these $\text{Sym}^d(P)$ are not stably birational to each other.

Let $d < e$ be different divisors of $i(P)$. There is thus a prime p such that $d = p^a d'$, $e = p^c e'$ where $a < c$ and d', e' are not divisible by p . Let p^b be the largest p -power dividing $i(P)$.

By assumption P has a k' point for some field extension k'/k of degree $i(P)$. Let k''/k be the Galois closure of k'/k and K the invariant subfield of a p -Sylow subgroup of $\text{Gal}(k''/k)$. Set $K' = k'K$. Note that p does not divide $\deg(K/k)$ and $\deg(K'/K) = p^b$, hence $i(P_K) = p^b$.

Although K'/K need not be Galois, the Galois group of its Galois closure is a p -group, hence nilpotent. Thus there is a subextension $K' \supset L \supset K$ of degree p^{b-a} . It is enough to show that $\text{Sym}^d(P_L)$ and $\text{Sym}^e(P_L)$ are not stably birational over L . By (1.4),

$$\text{Sym}^d(P_L) \stackrel{\text{stab}}{\sim} \text{Sym}^{p^a}(P_L) \quad \text{and} \quad \text{Sym}^e(P_L) \stackrel{\text{stab}}{\sim} \text{Sym}^{p^c}(P_L).$$

Note that P_L has a point in K' and $\deg(K'/L) = p^a$, hence $i(P_L) = p^a$ and so $\text{Sym}^{p^a}(P_L)$ is stably rational by Corollary 4. By contrast $\text{Sym}^{p^c}(P_L)$ does not have any L -points. Indeed, an L -point on $\text{Sym}^{p^c}(P_L)$ would mean a 0-cycle of degree p^c on P_L hence a 0-cycle of degree $p^{b-a}p^c = p^{b-a+c}$ on P_K . This is impossible since $i(P_K) = p^b$ and $b - a + c < b$. Thus $\text{Sym}^{p^a}(P_L)$ and $\text{Sym}^{p^c}(P_L)$ are not stably birational. \square

Remark 6. It is possible that the stable birational equivalences in Theorem 1 can be replaced by birational equivalences. For instance, it is possible that

$$\mathrm{Sym}^d(P) \stackrel{\mathrm{bir}}{\sim} \mathrm{Grass}(\mathbb{P}^{(d,i)-1}, P) \times \mathbb{P}^m \quad \text{for suitable } m.$$

However, several steps in the proof naturally give only stable birational equivalences and the difference between stable birational equivalence and birational equivalence is not even understood for Severi-Brauer varieties.

We have used some general results on symmetric powers.

Lemma 7. *Let U be a positive dimensional, geometrically irreducible k -variety. Then*

$$\mathrm{Sym}^m(U \times \mathbb{P}^r) \stackrel{\mathrm{bir}}{\sim} \mathrm{Sym}^m(U) \times \mathbb{P}^{rm}.$$

Proof. There is a natural projection map $\mathrm{Sym}^m(U \times \mathbb{P}^r) \rightarrow \mathrm{Sym}^m(U)$. We claim that its generic fiber F_{gen} is rational. To construct it, set

$$L := k(\mathrm{Sym}^m(U)) \quad \text{and} \quad K := k(\mathrm{Sym}^{m-1}(U) \times U).$$

Here we think of $\mathrm{Sym}^m(U)$ as U^m/S_m and $\mathrm{Sym}^{m-1}(U) \times U$ as U^m/S_{m-1} where $S_{m-1} \subset S_m$ are the permutations that fix the last factor. Thus K/L is a degree m field extension and $F_{\mathrm{gen}} \stackrel{\mathrm{bir}}{\sim} \mathfrak{R}_{K/L}(\mathbb{P}^r)$, the Weil restriction of \mathbb{P}^r from K to L . Thus F_{gen} is rational. \square

Corollary 8. *Let U, V be positive dimensional, geometrically irreducible k -varieties. If $U \stackrel{\mathrm{stab}}{\sim} V$ then $\mathrm{Sym}^m(U) \stackrel{\mathrm{stab}}{\sim} \mathrm{Sym}^m(V)$ for every m .* \square

As a consequence we see that $\mathrm{Sym}^m(\mathbb{P}^r)$ is stably rational. In fact it is rational; see [Mat68] for a very short proof.

2. MODULI OF SEVERI-BRAUER SUBVARIETIES

We need some results on twisted line bundles and maps between Severi-Brauer varieties.

Definition 9 (Twisted line bundles). Let X be a geometrically normal, proper k -variety. A *twisted line bundle* of X is a line bundle L on X_{k^s} such that $L^\sigma \cong L$ for every $\sigma \in \mathrm{Gal}(k^s/k)$. Equivalently, its class $[L]$ is a k -point of $\mathrm{Pic}(X)$. For example, if P is a Severi-Brauer variety then $\mathcal{O}_P(r)$ is a twisted line bundle for every r .

Let $|L|$ denote the irreducible component of the Hilbert scheme (or Chow variety) of X parametrizing subschemes $H \subset X$ such that H_{k^s} is in the linear system $|L_{k^s}|$. (See [Kol96, Chap.I.] for the Hilbert scheme or the Chow variety.) This is clearly a Severi-Brauer variety. There is a natural map $\iota_L : X \dashrightarrow |L|^\vee$ given by $x \mapsto \{H : H \ni x\}$.

Using this we define the *dual* of a Severi-Brauer variety P as $P^\vee := |\mathcal{O}_P(1)|$.

Let $\phi : X \dashrightarrow Y$ be a map between geometrically normal, proper varieties and L_Y a twisted line bundle on Y . Assume that either ϕ is a morphism or X is smooth. Then ϕ^*L_Y is a twisted line bundle on X and $|\phi^*L_Y| \sim |L_Y|$.

Let X, Y be geometrically normal, proper varieties and L_X, L_Y twisted line bundles on them. Let $\mathrm{Map}((X, L_X), (Y, L_Y))$ denote the moduli space of all maps $\phi : X \dashrightarrow Y$ such that $\phi^*L_Y \cong L_X$.

If P, Q are Severi-Brauer varieties then we write

$$\mathrm{Map}_d(Q, P) := \mathrm{Map}((Q, \mathcal{O}_Q(d)), (P, \mathcal{O}_P(1))).$$

Composing with ι_L gives an isomorphism

$$\mathrm{Map}((X, L), (P, \mathcal{O}_P(1))) \cong \mathrm{Map}_1(|L|^\vee, P).$$

10 (Severi-Brauer varieties II). Let P, Q be Severi-Brauer varieties.

- (1) Their product is defined as $|\mathcal{O}_{P^\vee \times Q^\vee}(1, 1)| \cong |\mathcal{O}_{P \times Q}(1, 1)|^\vee$. I denote this by $P \otimes Q$. It is better to think of this as defined on Brauer-equivalence classes. This makes the set of Brauer-equivalence classes into a group with identity $\mathbb{P}^0 \sim \mathbb{P}^m$ and inverse P^\vee . The group is torsion, more precisely $P^{\otimes i(P)} \sim \mathbb{P}^0$. (Frequently a smaller power of P is trivial, the smallest such exponent is the *period*.)
The group defined above is isomorphic to the *Brauer group* of k . (We will not use its cohomological description; see [GS06].)
- (2) $\mathrm{Map}_1(Q, P) \sim Q^\vee \otimes P$. The natural map is $\phi \mapsto \{(x, H) : \phi(x) \in H\} \in |\mathcal{O}_{Q \times P^\vee}(1, 1)|$.
- (3) If $|L|$ is non-empty then $|L^m| \sim |L|^{\otimes m}$; this comes from identifying the symmetric power of a vector space V with the subspace of symmetric tensors in $V^{\otimes m}$.
- (4) $\mathrm{Map}_d(Q, P) \sim (Q^\vee)^{\otimes d} \otimes P$; this follows from the previous two claims.
- (5) Again combining the previous two claims with (2.1) we conclude that there is a rational map $P \dashrightarrow Q$ iff Q is similar to $P^{\otimes m}$ for some m . (This is called Amitsur's theorem.)

We next define the spaces of Severi-Brauer subvarieties of a Severi-Brauer variety. That is, given a Severi-Brauer variety P we look at the subset of the Chow variety $\mathrm{Chow}(P)$ parametrizing subvarieties $X \subset P$ whose normalization \bar{X} is a Severi-Brauer variety. For technical reasons it is better to work with $\bar{X} \rightarrow P$.

Definition 11. Fix integers $0 \leq m \leq n$, $1 \leq d$ and a Severi-Brauer variety P of dimension n . Let $M_{\mathbb{P}^m}^\circ(P, d)$ denote the moduli space parametrizing morphisms $\phi : Q \rightarrow P$ satisfying the following assumptions.

- (1) Q is a Severi-Brauer variety of dimension m .
- (2) $\phi^* \mathcal{O}_P(1) \cong \mathcal{O}_Q(d)$.
- (3) Either $m < n$ and $\phi : Q \rightarrow \phi(Q)$ is birational or $m = n$ and every automorphism of the triple $(\phi : Q \rightarrow P)$ that is the identity on P is also the identity on Q .
- (4) Two such morphisms $\phi_i : Q_i \rightarrow P$ are identified if there is an isomorphism $\tau : Q_1 \cong Q_2$ such that $\phi_1 = \phi_2 \circ \tau$.

The spaces $M_{\mathbb{P}^m}^\circ(P, d)$ are quasi-projective. They should be thought of as open subschemes of the projective moduli spaces of stable maps $\bar{M}_{\mathbb{P}^m}(P, d)$ [Ale96]. Since we are interested in their birational properties, these compactifications are not important to us.

(Comment on the notation. The moduli space of maps from X to Y is frequently denoted by $\mathrm{Map}(X, Y)$. However, for moduli of stable maps from a genus g curve to Y the usual notation is $M_g(Y, \beta)$ where β is the homology class of the image. If $Y = \mathbb{P}^n$ then β is usually replaced by $\deg \beta$. Thus $M_{\mathbb{P}^m}^\circ(P, d)$ follows mostly the stable maps convention, except that the degree of $\phi(Q)$ is d^m .)

Note that if $\phi : Q \rightarrow \phi(Q)$ is birational then every automorphism of $\phi : Q \rightarrow P$ that is the identity on P is also the identity on Q . This is why the most naive way of identifying two maps is adequate in (4). (As we discussed earlier, failure of this is one of the problems with $\bar{M}_0(P, 2)$ if $\dim P = 1$.)

If $d = 1$ then we get $M_{\mathbb{P}^m}^\circ(P, 1) = \text{Grass}(\mathbb{P}^m, P)$ and if $m = 1$ then the $M_{\mathbb{P}^1}^\circ(P, d)$ are open subschemes of the space of genus 0 stable maps $\bar{M}_0(P, d)$.

These moduli spaces are closely related to the spaces of maps from Definition 9:

$$M_{\mathbb{P}^m}^\circ(P, d) \stackrel{\text{bir}}{\sim} \text{Map}_d(\mathbb{P}^m, P) / \text{Aut}(\mathbb{P}^m).$$

The resulting map $\Pi : \text{Map}_d(\mathbb{P}^m, P) \dashrightarrow M_{\mathbb{P}^m}^\circ(P, d)$ is not a product, not even birationally. Indeed the fiber of Π over a given $\phi : Q \rightarrow P$ is the space of isomorphisms $\text{Isom}(\mathbb{P}^m, Q)$. This is a principal homogeneous space under $\text{Aut}(\mathbb{P}^m)$ but it is not isomorphic to $\text{Aut}(\mathbb{P}^m)$ unless Q is trivial.

Our aim is to understand the spaces $M_{\mathbb{P}^m}^\circ(P, d)$ for arbitrary ground fields. This is achieved only for $m = 1$ but we have the following general periodicity property.

Theorem 12. *Let $P \sim P'$ be Brauer equivalent Severi-Brauer varieties of dimensions n, n' . Fix $0 \leq m \leq \min\{n, n'\}$ and $1 \leq d, d'$. Assume that $d \equiv d' \pmod{m+1}$. Then*

$$M_{\mathbb{P}^m}^\circ(P, d) \stackrel{\text{stab}}{\sim} M_{\mathbb{P}^m}^\circ(P', d').$$

Proof. The idea is similar to the “no-name method” explained in [Dol87, Sec.4], where it is attributed to Bogomolov and Lenstra.

Let $\text{Isom}_{\mathbb{P}^m}(d, P, d', P')$ denote the scheme parametrizing triples

$$\{(\phi : Q \rightarrow P); (\phi' : Q' \rightarrow P'); \tau\}$$

where $(\phi : Q \rightarrow P) \in M_{\mathbb{P}^m}^\circ(P, d)$, $(\phi' : Q' \rightarrow P') \in M_{\mathbb{P}^m}^\circ(P', d')$, and $\tau : Q \rightarrow Q'$ is an isomorphism. (No further assumptions on ϕ and $\phi' \circ \tau$.) We prove that

$$M_{\mathbb{P}^m}^\circ(P, d) \stackrel{\text{stab}}{\sim} \text{Isom}_{\mathbb{P}^m}(d, P, d', P') \stackrel{\text{stab}}{\sim} M_{\mathbb{P}^m}^\circ(P', d'),$$

using the natural projections

$$\pi : \text{Isom}_{\mathbb{P}^m}(d, P, d', P') \rightarrow M_{\mathbb{P}^m}^\circ(P, d) \quad \text{and} \quad \pi' : \text{Isom}_{\mathbb{P}^m}(d, P, d', P') \rightarrow M_{\mathbb{P}^m}^\circ(P', d').$$

It is sufficient to show that their generic fibers are rational. The roles of d, d' are symmetrical, thus it is enough to consider $\pi : \text{Isom}_{\mathbb{P}^m}(d, P, d', P') \rightarrow M_{\mathbb{P}^m}^\circ(P, d)$.

Note that the fiber of π over $(\phi : Q \rightarrow P)$ consists of pairs

$$\{(\phi' : Q' \rightarrow P'); \tau\}$$

where $(\phi' : Q' \rightarrow P') \in M_{\mathbb{P}^m}^\circ(P', d')$ and $\tau : Q \rightarrow Q'$ is an isomorphism. Specifying such a pair is the same as giving $(\phi \circ \tau : Q \rightarrow P') \in \text{Map}_d(Q, P')$. Thus the fiber of π over $(\phi : Q \rightarrow P)$ is isomorphic to $\text{Map}_{d'}(Q, P')$.

Let K be the function field of $M_{\mathbb{P}^m}^\circ(P, d)$. We thus have a morphism

$$\phi_K : Q_K \rightarrow P_K \quad \text{such that} \quad \phi_K^* \mathcal{O}_{P_K}(1) \cong \mathcal{O}_{Q_K}(d).$$

By (10.4) ϕ_K corresponds to a K -point of $\text{Map}_d(Q_K, P_K) \sim (Q_K^\vee)^{\otimes d} \otimes P_K$. Thus $(Q_K^\vee)^{\otimes d} \otimes P_K$ is rational by (2.1). Furthermore, since $d \equiv d' \pmod{m+1}$, we know that $Q_K^{\otimes d'} \sim Q_K^{\otimes d}$ by (10.1), hence $(Q_K^\vee)^{\otimes d'} \otimes P'_K$ is stably rational by (10.1), hence in fact rational by (2.1). Therefore

$$\text{Map}_{d'}(Q_K, P'_K) \sim (Q_K^\vee)^{\otimes d'} \otimes P'_K$$

is also rational by (2.1). \square

Remark 13. There are a few more cases when one can get stable birational equivalences. For example, assume that d, d' and $\text{per}(P) = \text{per}(P')$, the period of P , are all relatively prime to $m + 1$. Then

$$\phi_K^* \mathcal{O}_{P_K}(\text{per}(P)) \cong \mathcal{O}_{Q_K}(d' \cdot \text{per}(P))$$

implies that Q_K is trivial. Using this observation for $d' = 1$ we obtain that

$$M_{\mathbb{P}^m}^\circ(P, d) \stackrel{\text{stab}}{\sim} M_{\mathbb{P}^m}^\circ(P', 1) \cong \text{Grass}(\mathbb{P}^m, P') \quad \text{if} \quad (m + 1, d \cdot \text{per}(P)) = 1.$$

As a consequence of Theorem 12, in order to describe the stable birational types of $M_{\mathbb{P}^m}^\circ(P, d)$, it is sufficient to understand $M_{\mathbb{P}^m}^\circ(P, d)$ for $d \leq m + 1$. There are two cases for which the answer is easy to derive.

Lemma 14. *Let P be a Severi-Brauer variety. Then*

- (1) $M_{\mathbb{P}^m}^\circ(P, d) \stackrel{\text{stab}}{\sim} \text{Grass}(\mathbb{P}^m, P)$ if $d \equiv 1 \pmod{m + 1}$.
- (2) $M_{\mathbb{P}^m}^\circ(P, d) \stackrel{\text{stab}}{\sim} P$ if $d \equiv 0 \pmod{m + 1}$ and $(m + 1) \mid 420$.

Proof. If $d \equiv 1 \pmod{m + 1}$ then $M_{\mathbb{P}^m}^\circ(P, d) \stackrel{\text{stab}}{\sim} M_{\mathbb{P}^m}^\circ(P, 1)$ by Theorem 12 and, essentially by definition, $M_{\mathbb{P}^m}^\circ(P, 1) = \text{Grass}(\mathbb{P}^m, P)$.

For the second claim we check the stable birational isomorphisms

$$M_{\mathbb{P}^m}^\circ(P, m + 1) \stackrel{\text{stab}}{\sim} M_{\mathbb{P}^m}^\circ(P, m + 1) \times P \stackrel{\text{stab}}{\sim} P.$$

First let K be the function field of $M_{\mathbb{P}^m}^\circ(P, m + 1)$. We need to show that P_K is trivial. By assumption, there is a K -map in $\text{Map}_{m+1}(Q_K, P_K)$ where $\dim Q_K = m$. By (10.4) this corresponds to a K -point of $(Q_K^\vee)^{\otimes m+1} \otimes P_K$. By (10.1) $Q_K^{\otimes m+1}$ is trivial, so $\text{Map}_{m+1}(Q_K, P_K) \sim P_K$ and so P_K is trivial.

For the second part, let L be the function field of P . Then P_L is trivial, hence

$$M_{\mathbb{P}^m}^\circ(P_L, m + 1) \cong M_{\mathbb{P}^m}^\circ(\mathbb{P}_L^m, m + 1) \stackrel{\text{bir}}{\sim} \mathbb{P}(H^0(\mathbb{P}_L^m, \mathcal{O}_{\mathbb{P}^m}(m + 1))^{n+1}) / \text{PGL}_{m+1}.$$

It is conjectured that this quotient is always stably rational, but this seems to be known only when $(m + 1) \mid 420$; see [For02, p.316] and the references there. \square

We give a geometric proof that the space of conics $\bar{M}_0(P, 2)$ is stably birational to P for $\dim P \geq 2$.

15 (Conics in Severi-Brauer varieties). We compute, in 2 different ways, the space T parametrizing triples (C, ℓ_1, ℓ_2) where the ℓ_i are secant lines of C .

Forgetting the lines gives a map to $\bar{M}_0(P, 2)$. Let C_K be the conic corresponding to the generic point of $\bar{M}_0(P, 2)$. A secant line of C_K is determined by $\text{Sym}^2 C_K \cong \mathbb{P}_K^2$. Thus $T \stackrel{\text{bir}}{\sim} \bar{M}_0(P, 2) \times \mathbb{P}^4$.

The secants lines ℓ_i meet at a unique point; this gives a map $T \rightarrow P$. Given $p \in P$, the fiber is obtained by first picking 2 points in $\mathbb{P}(T_p P) \cong \mathbb{P}_{k(p)}^{n-1}$. Once we have 2 lines, they determine a plane $\langle \ell_1, \ell_2 \rangle$ and the 5-dimensional linear system $|\ell_1 + \ell_2|$ on the plane $\langle \ell_1, \ell_2 \rangle$ gives the conics. Thus $T \stackrel{\text{bir}}{\sim} P \times \mathbb{P}^{2n+3}$ and hence $\bar{M}_0(P, 2) \stackrel{\text{stab}}{\sim} P$.

16 (Proof of Theorem 1.7–8). If $d = 2e$ is even then

$$\bar{M}_0(P, 2e) \stackrel{\text{bir}}{\sim} M_{\mathbb{P}^1}(P, 2e) \stackrel{\text{stab}}{\sim} M_{\mathbb{P}^1}(P, 2) \stackrel{\text{bir}}{\sim} \bar{M}_0(P, 2),$$

where the birational equivalences are by definition and the stable birational equivalence holds by Theorem 12. Next $\bar{M}_0(P, 2) \stackrel{\text{stab}}{\sim} P$ follows either from Lemma 14.2 or from Paragraph 15. This gives (1.7).

Similarly, if $d = 2e + 1$ is odd then (1.8) follows from

$$\bar{M}_0(P, 2e + 1) \stackrel{\text{bir}}{\sim} M_{\mathbb{P}^1}(P, 2e + 1) \stackrel{\text{stab}}{\sim} M_{\mathbb{P}^1}(P, 1) = \text{Grass}(\mathbb{P}^1, P). \quad \square$$

Remark 17. So far we have worked with a fixed Severi-Brauer variety P , but it would be interesting to understand how the $\text{MSym}(P)$ for different Severi-Brauer varieties interact with each other.

For example, assume that P, Q are Severi-Brauer varieties such that $\text{index}(P)$ and $\text{index}(Q)$ are relatively prime. We claim that $\text{Sym}^d(P)$ and $\text{Sym}^e(Q)$ are stably birational to each other iff they are both stably rational.

To see this, assume that $\text{Sym}^e(Q)$ is not stably rational. By Corollary 4 this holds iff $\text{index}(Q) \nmid e$. Set $K := k(P)$. Since $\text{index}(P)$ and $\text{index}(Q)$ are relatively prime, $\text{index}(Q_K) = \text{index}(Q)$, thus $\text{Sym}^e(Q_K)$ is not stably rational by Corollary 4. By contrast, P_K is trivial hence $\text{Sym}^d(P_K)$ is stably rational; even rational by [Mat68]. Thus $\text{Sym}^d(P)$ and $\text{Sym}^e(Q)$ are not stably birational to each other.

Further steps in this direction are in Theorem 12; see also [Kol05, Hog09] for related questions.

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